Chapter 2

The continuum hypothesis

2.1 The continuum hypothesis

In order to arrive at the equations governing the static equilibrium and the motion of a fluid, as well as the distribution of its properties, we have to apply the same fundamental laws of physics found for the dynamics of a point mass. The essential difference is that now we have to manage not isolated point masses, but a continuous medium, that is, matter that fills space with continuity. While the various properties of the point masses (momentum, kinetic energy, and so on) are associated to the points where they are situated, now the various properties (velocity, density, temperature, and so on) are to be defined in all the points lying within the domain occupied by the fluid. So, a fluid is described by the *fields* of the various quantities that define its properties. They can be *scalar fields*, that is, described by a scalar quantity, such as temperature, or *vector fields*, when the involved quantity is a vector, such as velocity.

Moreover, a fluid is described as a *continuum*, that is, we assume that every particle of fluid can be indefinitely subdivided into smaller and smaller particles, and that each quantity involved tends to become constant over the smaller particles as their dimensions tend to zero. In other words, if a certain quantity is not constant over a given particle, we assume that the variations of that quantity over its parts are progressively reduced as these parts decrease. Thus any quantity can be described by means of continuous functions of the geometrical position in space.

This does not prevent the possible presence of one or more discontinuity surfaces, across which various quantities can change abruptly. For example, the free surface of a liquid separates the very different properties of the liquid and the air. Similarly, two liquids of different density are separated by a well-defined interface. However, inside each layer of fluid the above mentioned principle holds, and the various fields are continuous. In other words, the continuum hypothesis implies that the number of possible surfaces of discontinuity must be finite.

Furthermore, it is assumed that changes in time of a property are gradual, and cannot occur suddenly. This means that the functions representing physical quantities are also continuous in time.

Sometimes, it is convenient to assume a discontinuity of a process in time as well. In this case we usually assume the presence of a single discontinuity.

A particle of fluid of infinitesimal size will be denoted by the term *parcel*. Thus, we can see the fluid as a set of many neighboring parcels within which the various quantities are constant or vary linearly. The equilibrium, motion and every other property of the fluid can be tracked back to the interaction among the various parcels and between the parcels of the fluid and the external environment. Strictly speaking this is not true. However small the parcels are defined, over time the different motion of their parts causes a certain deformation of their geometry, ultimately ending, often, in very complicated structures impossible to manage. Nevertheless, the representation of the fluid as the sum of many adjacent parcels of simple geometry is essentially true for limited intervals of time, and leads to a good understanding of the behavior of the fluid as a whole.

In the simplest flows, such as uniform and circular flows, the parcel can maintain the same shape for a long interval of time, moving as a rigid body. In other words, its motion can be described as the combination of a translation and a rotation. In the most part of flows, however, the parcel undergoes a deformation. It can be stretched or shrunk in the various directions, or subjected to a change of the angles between its faces. If the fluid is compressible, this deformation can be accompanied by a variation of volume. In the most irregular flows the deformation is very strong and leads to the formation of spiral-shaped tentacles that extend over increasingly larger regions of space, while confusing with the similar structures of the other neighboring parcels.

2.2 The decoupling between molecular and macroscopic scales

The representation of the fluid as a continuum is in contrast with what we know today about the molecular structure of matter. For a single molecule, or a small cluster of molecules, many macroscopic quantities loose their meaning. For example, to speak about *the pressure of a molecule* is meaningless. But it has been verified that the smallest scales of the motions occurring in nature are much larger than the molecular ones. The smallest length scales are of the order of hundredths of millimeter, and the shortest time scales are of the order of hundredths of second. Thus, from a practical point of view, geophysical fluids can be considered as a continuum, without any appreciable error. This statement can be extended to the laboratory as well, with a few exceptions, among which the case of very rarefied gases.

A kilomole of air at a temperature of 0°C and at a pressure of 1 atmosphere occupies 22.4 m^3. It contains a number of molecules equal to Avogadro's number $N_A = 6.022 \times 10^{26}$ kmol^{-1}. Hence, the number of air molecules per cubic meter is about 2.7×10^{25}, equivalent to 27 millions in a cube with a side equal to 1 μm. This number means that, on average, along each edge of the cube there are about 300 molecules. The *mean free path*, that is, the distance covered on average by a molecule before colliding with another molecule, is of the order of 10^{-7} m = 0.1 μm. In a liquid the molecular density is about three orders of magnitude higher. Under these conditions it is apparent that a parcel with a side of a few μm is already a good approximation of an infinitesimal parcel in which all the properties are statistically uniform in space, as required by the continuum hypothesis.

The average time between the collision between two atmospheric molecules is of the order of 10^{-10} s, which is much shorter than the smallest time scale of a natural or artificially produced phenomenon. Thus we can consider the quantities describing a parcel uniform in time as well.

When a gas is so rarefied that the mean free path of its molecules is of the same order of magnitude as the typical dimensions of the container, then the continuum hypothesis becomes meaningless. These problems must be studied by means of different tools.

For liquids the distance between molecules is much smaller than for gas, so that the uniformity in space of a quantity is even more guaranteed. An accurate evaluation of the characteristic times of the molecular interactions is not possible, but they are in any case very small, so that the continuity in time is also guaranteed.

In other words, we observe a decoupling between molecular and macroscopic motions. This is a very important feature of the fluid motions, because this means that they can be studied without any reference to the molecular theory.

In the following, when this will be possible, we will provide the qualitative molecular interpretations of the various phenomena, because often they are very simple.

However, we must not forget that a parcel is formed by molecules. This means that in the neighborhood of the surface of a parcel there is a continuous exchange of molecules across it, due to thermal agitation. This process implies on one hand a loss of molecules from the original parcel and on the other hand a gain of a statistically equal number of molecules from the external fluid. The intensity of the process is proportional to the extension of the surface of the parcel. Thus, it is quicker when the motion is highly irregular with strong variations of velocity over small space scales, because this leads to a strong deformation of the shape of the parcel with a consequent growth of its surface. But the process is also present in slow motions, even in a fluid at rest. At the end, in any case, the identity of the original parcel is completely lost.

The molecules that initially formed the parcel, with time are scattered more or less uniformly over the whole volume occupied by the fluid.